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ON A DISCRETE TIME APPROXIMATION OF THE HAMILTON-JACOBI EQUATION OF DYNAMIC PROGRAMMING

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ON A DISCRETE TIME APPROXIMATION OF THE HAMILTON-JACOBI EQUATION OF DYNAMIC PROGRAMMING

SUR UNE APPROXIMATION EN TEMPS DISCRET DE L'EQUATION D'HAMILTON-JACOBI DE LA PROGRAMMATION DYNAMIQUE

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RESUME

Dans ce travail nous considérons des problèmes de contrôle optimal des systèmes gouvernés par des équations différentiels ordinaires. Nous faisons l'analyse de une approximation discrétisée dans le temps et nous étudions la vitesse de convergence de la solution discrétisée à la solution du problème original. Nous avons prouvé, en employant des techniques de l'analyse convexe, que la vitesse de convergence est de l'ordre $h^{(\gamma/2)\wedge(1/2)}$ dans le cas général (où $\gamma\in\Re^+$ dépend des données du problème), et de l'ordre $h^{(\gamma/2)\wedge 1}$ dans le cas où des hypothèses de semiconcavité sont valables.

ABSTRACT

In this paper we consider optimal control problem of systems described by ordinary differential equations. We analyze its discrete time approximation and we study the rate of convergence of the approximate solutions to the solution of the original problem. We prove using convex analysis techniques that the rate is of order $h^{(\gamma/2)\wedge(1/2)}$ in the general case (where $\gamma\in\Re^+$ is a constant depending on the problem data), and of order $h^{(\gamma/2)\wedge 1}$ when semiconcavity hypotheses hold .

CONTENTS

1. INTRODUCTION

Dynamic programming theory prove that the value function of an optimal control problem for systems described by ordinary differential equations satisfies (provided smooth regularity conditions hold) a nonlinear partial differential equation of first order of type Hamilton-Jacobi-Bellman (see [4]). Classical procedures cannot be employed directly because even in simple problems the optimal value function has discontinuous partial derivatives. To avoid these nuisances it can be considered (see [3] and [7]) that the optimal value function is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation associated to those optimal control problems which for the particular case of infinite horizon takes the form:

$$\min_{\alpha\in A} \left\{ -\lambda u + \sum_{i=1}^{\nu} g_i^\alpha \frac{\partial u}{\partial x_i} + f^\alpha \right\} = 0, \text{ in } \Re^\nu, \ \ \lambda \in \Re^+, \tag{1}$$

In this paper we study the properties of the family of functions u^h, discrete time approximations of Hamilton-Jacobi-Bellman equations. We obtain explicit estimates of the rate of convergence of these approximations to the viscosity solution of the original problem.

We will also deal with the optimal control problem with finite horizon. The corresponding equation takes the form:

$$\min_{\alpha\in A} \left\{ \frac{\partial u_T}{\partial t} - \lambda u_T + \sum_{i=1}^{\nu} g_i^\alpha \frac{\partial u_T}{\partial x_i} + f^\alpha \right\} = 0, \ \ \text{a.e. } (t,x)\in [0,\ T]\times\Re^\nu \tag{2}$$

$$u_T(T,\ x) = 0 \ \ \forall\ x \in \Re^\nu \tag{3}$$

The central results are the following:

- Using convex analysis techniques and introducing a suitable family of finite horizon problems, we obtain the following error estimates for the finite horizon problems:

$$\boxed{\left| u_T(t,\ x) - u_T^h(t,\ x) \right| \le M\,\phi(T-t)\ h^{1/2}} \tag{4}$$

and when semiconcavity assupmtions hold for f,g:

$$\boxed{\left| u_T(t,x) - u_T^h(t,x) \right| \leq M\, \psi(T-t)\, h} \tag{5}$$

• Based on estimations (4) and (5), an argument using a time optimization procedure gives for the infinite horizon problem the estimations (6), valid for the general case and (7) valid for the case with semiconcavity assumptions:

$$\boxed{\left| u(x) - u^h(x) \right| \leq C\, h^{(\gamma/2)\wedge(1/2)}} \tag{6}$$

$$\boxed{\left| u(x) - u^h(x) \right| \leq C h^{1\wedge(\gamma/2)}} \tag{7}$$

where γ, ϕ and ψ are functions of the problem data.

These results are closely related to those obtained in [2] and [8]. The most remarkable differences are the use of convex analysis techniques to prove them, and that the estimates obtained using semiconcavity assumptions are sharper (in the set $1<\gamma<2$) than those presented in [2].

2. DESCRIPTION OF THE PROBLEM

2.1 Description of the Problem with Infinite Horizon

2.1.1 *Definition of the optimal cost function* u

The problem consists in to find the optimal value function u, such that:

$$u(x) = \inf_{\alpha \in \mathcal{A}} J(x,\alpha(\cdot)) \quad \forall x \in \Re^{\nu}. \tag{8}$$

where $\mathcal{A}$ is the set of measurable functions $\alpha(\cdot)$ defined in $[0,\infty)$ with values in a compact subset $A \subset \Re^m$,

$$J(x,\alpha(\cdot)) = \int_0^{\infty} f(y(x,s),\alpha(s))\, e^{-\lambda s}\, ds$$

$f : \Re^{\nu} x\ A \rightarrow \Re$ is the instantaneous cost and $\lambda > 0$, is the discount rate.

The state of the system $y(s)=y(x,s)$ corresponding to the control $\alpha(\cdot)$ and to the initial condition x, is given by the following differential equation ($g : \Re^{\nu} x\ A \rightarrow \Re^{\nu}$):

$$\dot{y}(s) = g(y(s),\, \alpha(s)), \quad s>0, \tag{9}$$

$$y(0) = x.$$

It is well known (see [4]) that equation (1) has not, in general a C^1 solution, so we consider the viscosity solution of Hamilton-Jacobi-Bellman equation. A function u is called a viscosity solution of equation (1) iff for any function $\phi \in C^1(\Re^{\nu})$ it is verified that:

(i) If $u-\phi$ has a local maximum in x_0, then:

$$\min_{\alpha \in A} \left\{ -\lambda u + \sum_{i=1}^{\nu} g_i^{\alpha} \frac{\partial \phi}{\partial x_i} + f^{\alpha} \right\} \leq 0, \text{ in } x_0.$$

(ii) If $u-\phi$ has a local minimum in x_1, then:

$$\min_{\alpha \in A} \left\{ -\lambda u + \sum_{i=1}^{\nu} g_i^{\alpha} \frac{\partial \phi}{\partial x_i} + f^{\alpha} \right\} \geq 0, \text{ in } x_1.$$

The function u, although is not continuously differentiable, is bounded and Hölder continuous as it is established in the following section.

2.1.2 *Properties of the optimal cost function* u

Proposition 2.1: *If the following inequalities are verified* $\forall x,\ \hat{x} \in \Re^{\nu},\ \alpha \in A$.

$$\begin{aligned} \| g(x,\alpha) - g(\hat{x},\alpha) \| &\le L_g \| x - \hat{x} \|, \\ \| g(x,\alpha) \| &\le M_g, \end{aligned} \tag{10}$$

$$\begin{aligned} | f(x,\alpha) - f(\hat{x},\alpha) | &\le L_f \| x - \hat{x} \|, \\ | f(x,\alpha) | &\le M_f, \end{aligned} \tag{11}$$

then u *satisfies:*

$$\begin{aligned} | u(x) | &\le \frac{M_f}{\lambda}, \\ | u(x) - u(\hat{x}) | &\le C \| x - \hat{x} \|^{\gamma}, \end{aligned} \tag{12}$$

$\forall x,\ \hat{x} \in \Re^{\nu}$, *where*

$$C = \begin{cases} \dfrac{L_f}{\lambda - L_g} & \gamma = 1 \qquad \text{if } \lambda > L_g, \\[2ex] L_f^{\gamma}\ (2\,M_f)^{1-\gamma} \left(\dfrac{1}{\lambda} + \dfrac{1}{L_g - \lambda} \right) & \gamma = \dfrac{\lambda}{L_g} \qquad \text{if } \lambda < L_g, \\[2ex] \left(\dfrac{2\,M_f}{L_f} \right)^{1-\gamma} \dfrac{1}{1-\gamma}\, e^{-\gamma} & \gamma \in (0,1) \qquad \text{if } \lambda = L_g. \end{cases}$$

Moreover, u *is the unique viscosity solution of Hamilton-Jacobi-Bellman equation* (1) (*see* [7]).

2.2 Description and Characteristics of the Problem with Finite Horizon

2.2.1 *The optimal cost function for the finite horizon case*

In this case, for the finite horizon problem, the optimal value function u_T is defined by:

$$u_T(t, x) = \inf_{\alpha \in \mathcal{A}_T} J_T(t, x, \alpha(\cdot)) \quad \forall\ t \in [0, T],\ \forall\ x \in \Re^{\nu} \tag{13}$$

where $\mathcal{A}_T$ is the set of measurable functions defined in [0,T) with values in a compact subset $A \subset \Re^m$,

$$J_T(t, x, \alpha(\cdot)) = \int_t^T f(y(x,s),\alpha(s))\, e^{-\lambda s}\, ds \tag{14}$$

and $y(\cdot)$ is the solution of (9) with initial conditions

$$y(t) = x \tag{15}$$

2.2.2 *Properties of the optimal cost function* u_T

In this case, it is possible to prove that function u_T is always bounded and Lipschitz continuous.

Proposition 2.2: *If conditions* (10) *and* (11) *are satisfied, then:*

$$|\, u_T(t, x)\, | \leq \frac{M_f}{\lambda}\,(1 - e^{-\lambda(T-t)}) \tag{16}$$

$$|\, u_T(t, x) - u_T(t, \hat{x})\, | \leq \begin{cases} \dfrac{L_f}{L_g - \lambda}\,(e^{(L_g - \lambda)(T-t)} - 1)\, \| x - \hat{x} \| & \textit{if } L_g \neq \lambda \\[2ex] L_f\,(T - nh)\, \| x - \hat{x} \| & \textit{if } L_g = \lambda \end{cases} \tag{17}$$

The viscosity solution of equation (2) is defined in a similar way as it has been done in the case of infinite horizon (see [3]). Function u_T is the unique solution of (2) in the viscosity sense.

3. APPROXIMATION OF HAMILTON-JACOBI-BELLMAN EQUATIONS

3.1 Analysis of the Problem with Infinite Horizon

3.1.1 *A discrete time scheme of approximation*

To find a time-approximated solution of (1), we consider the equation:

$$\max_{\alpha\in A} \left\{ u^h(x) - (1-\lambda h)\, u^h(x+hg(x,\alpha)) - h f(x,\alpha) \right\} = 0, \quad x\in\Re^\nu,\ \lambda>0. \tag{18}$$

We can prove that if $0<h<\frac{1}{\lambda}$, then (18) has a unique bounded uniformly continuous solution u^h; in fact, they are equi-uniformly Hölder continuous functions and $\{u^h\}$ converges uniformly in $\Re^\nu$, when h tends to zero, to the unique viscosity solution of (1).

Moreover, $\forall x\in\Re^\nu$ (see [1] and [2]) it can be proved that:

$$u^h(x) = \min_{\alpha\in\mathcal{A}^h} J^h(x,\alpha) \tag{19}$$

where $\mathcal{A}^h$ denotes the subset of $\mathcal{A}$ of the controls that have a constant value in the interval $[\,kh, (k+1)h\,)$, $k = 0, 1, \ldots$

$$J^h(x,\alpha) = h \sum_{k=0}^{\infty} f\Big(y_h(x,k),\alpha(kh)\Big)\,\Big(1-\lambda h\Big)^k$$

and the sequence $y_h(x,k)$ is given by the following recursive formula:

$$y_h(x,k+1) = y_h(x,k) + h\, g\Big(y_h(x,k),\alpha(kh)\Big), \quad k = 0,1,\ldots \tag{20}$$

$$y_h(x,0) = x,$$

3.1.2 *Properties of the function* u^h

The discrete time solution u^h, also as the continuous function u, is bounded and Hölder continuous.

Proposition 3.1: *If* (10) *and* (11) *are valid, then:*

$$\left| u^h(x) \right| \le \frac{M_f}{\lambda}$$

$$\left| u^h(x) - u^h(\hat{x}) \right| \le C \| x - \hat{x} \|^{\gamma},$$

where $h \in (0,\frac{1}{\lambda})$, $\forall x, \hat{x} \in \Re^{\nu}$

$$C = \left| \begin{array}{lll} \dfrac{L_f}{\lambda - L_g} & \gamma = 1 & \text{if } \lambda > L_g, \\ \\ L_f^{\gamma} (2M_f)^{1-\gamma} \left(\dfrac{1}{\lambda} + \dfrac{1}{L_g - \lambda} \right) & \gamma = \dfrac{\lambda}{L_g} & \text{if } \lambda < L_g, \\ \\ \left(\dfrac{2M_f}{L_f} \right)^{1-\gamma} \dfrac{1}{1-\gamma} e^{-\gamma} & \gamma \in (0,1) & \text{if } \lambda = L_g. \end{array} \right.$$

3.2 Analysis of the Problem with Finite Horizon

3.2.1 *The recursive discrete time scheme of approximation*

In this case we define recursively function $u_T^h(n, x)$, for $h = \frac{T}{\mu}$, $\mu \in \mathcal{N}^+$, $n = 0, \ldots, \mu$.

$$u_T^h(n-1, x) = \min_{\alpha \in A} \left\{ (1 - \lambda h)\, u_T^h\Big(n, x + hg(x,\alpha)\Big) + h f(x,\alpha) \right\} \quad n = 1, \ldots, \mu \tag{21}$$

$$u_T^h(\mu, x) = 0 \quad \forall\, x \in \Re^{\nu} \tag{22}$$

It is obvious that u_T^h is the solution of the following optimization problem:

$$u_T^h(n, x) = \min_{\alpha \in \mathcal{A}_T^h} J_T^h(n, x, \alpha(\cdot)) \tag{23}$$

where $\mathcal{A}_T^h$ denotes the subset of $\mathcal{A}_T$ of the controls that have a constant value in the interval $[kh, (k+1)h)$, $k = 0, 1, \ldots, (\mu - 1)$.

$$J_T^h(n, x, \alpha(\cdot)) = h \sum_{k=n}^{\mu-1} f\Big(y_h(x,k), \alpha(kh)\Big) \Big(1 - \lambda h\Big)^{k-n} \tag{24}$$

The sequence $y_h(x,k)$ is given by the following recursive formula:

$$y_h(x,k+1) = y_h(x,k) + h\, g\Big(y_h(x,k),\alpha(kh)\Big), \qquad k=n,\ldots,(\mu-1). \tag{25}$$

$$y_h(x,n)=x$$

3.2.2 *Properties of the function* u^h_T

The following proposition establishes some important regularity properties of function u^h_T; in particular, the fact that it is always Lipschitz continuous plays a key rôle in the proofs of the central results of this paper.

Proposition 3.2: u^h_T *is uniformly bounded and Lipschitz continuous; i.e.* $\forall\ h \in (0,\frac{1}{\lambda})$, $\forall x, \hat{x} \in \Re^\nu$

$$\left| u^h_T(n,x) \right| \le M_f \frac{(1-(1-\lambda h)^{\mu-n})}{\lambda} \tag{26}$$

$$\left| u^h_T(n,x) - u^h_T(n,\hat{x}) \right| \le L_{u^h_T}(n)\, \| x - \hat{x} \| \tag{27}$$

$$L_{u^h_T}(n) \le \begin{cases} L_f \dfrac{1}{L_g-\lambda}\, e^{(L_g-\lambda)(T-nh)} & \text{if } L_g > \lambda \\[2ex] L_f \dfrac{1}{\lambda-L_g} & \text{if } L_g < \lambda \\[2ex] L_f\,(T-nh) & \text{if } L_g = \lambda \end{cases}$$

Proof: By (11), (23) and (24) it follows easily the validity of (26). It is clear too that for $n = \mu$, u^h_μ is Lipschitz continuous with $L_{u^h_\mu} = 0$, because $u^h_\mu \equiv 0$

To complete the induction procedure, we consider the following inequality:

$$u^h_T(n,x) - u^h_T(n,y) \le (1-\lambda h)\, u^h_T\Big(n+1,x+h\,g(x,\tilde{a})\Big)+h\,f(x,\tilde{a}) - (1-\lambda h)\, u^h_T\Big(n+1,y+h\,g(y,\tilde{a})\Big)-h\,f(y,\tilde{a})$$

where $\tilde{a}$ realizes the minimum of (21) for $u^h_T(n,y)$, then

$$u^h_T(n,x) - u^h_T(n,y) \le \Big((1-\lambda h)(1+L_g h)\, L_{u^h_T}(n+1)+ L_f\, h\Big) \| x - y \|$$

analogously for $u^h_T(n,y) - u^h_T(n,x)$, then we obtain:

$$L_{u_T^h}(n) \le (1-\lambda h)(1+L_g h)\, L_{u_T^h}(n+1) + L_f\, h \tag{28}$$

Analysis of different cases

• **Case $L_g = \lambda$**

In this case we have that $(1-\lambda h)(1+L_g h) = 1-(\lambda h)^2 < 1$, then:

$$L_{u_T^h}(n) \le L_{u_T^h}(n+1) + L_f\, h$$

which implies

$$L_{u_T^h}(n) \le L_{u_T^h}(\mu) + (\mu - n)\, L_f\, h = L_f\,(T - nh)$$

• **Case $L_g < \lambda$**

The formula (28) becomes

$$L_{u_T^h}(n) \le \Big(1 - (\lambda - L_g)h \Big)\, L_{u_T^h}(n+1) + L_f\, h$$

then

$$L_{u_T^h}(n) \le L_f h\, \frac{1 - \Big(1 - (\lambda - L_g)h\Big)^{\mu - n}}{h\,(\lambda - L_g)} \le L_f\, \frac{1}{\lambda - L_g}$$

• **Case $L_g > \lambda$**

The inequality (28) becomes

$$L_{u_T^h}(n) \le L_f h\, \frac{\Big(1 + (L_g - \lambda)h \Big)^{\mu - n}}{h\,(L_g - \lambda)}$$

so

$$L_{u_T^h}(n) \le L_f\, \frac{1}{L_g - \lambda}\, e^{(L_g - \lambda)(T - nh)}$$

□

Remark 3.1: From here we will denote with L_u the Lipschitz constant of u_T^h and we will discriminate the three cases between them when it were necessary.

Remark 3.2: From here, and in order to obtain simplicity of notation and clarity of arguments, we will use letters C, M, K to denote arbitrary constants (which values depends on the context where they appear) which depend on the data of the problem (constants λ, M_g, M_f, L_g, L_f, etc.) but do not depend on the parameter of discretization h, of the regularization ρ, etc.

4. APPROXIMATION OF CONTROL POLICIES WITH STEP-FUNCTIONS

4.1 A Convexity Result

Proposition 4.1: *Let* f: $A \rightarrow \Re^r$ *be a map with* $F=\{f(\alpha)/\alpha \in A\}$ *a compact set, then*

$$w_0 \in \mathrm{Co}\,F \tag{29}$$

where $w_0=\int_0^1 f(\alpha(t))\,dt$, *and* Co F *is the convex enveloppe of* F, *i.e.:*

$$\mathrm{Co}\,F = \left\{ \sum_{i=1}^{n} \lambda_i x_i \ , \ \lambda_i \geq 0 \ , \ \sum_{i=1}^{n} \lambda_i = 1 \ , \ x_i \in F, \ n \in \mathcal{N} \right\}$$

Proof: The proof consists basically in the establishment of the closeness of Co F.
We know (see [9]) that in a finite dimensional space (dimension r in this case) we have:

$$\mathrm{Co}F = \left\{ \sum_{i=1}^{r+1} \lambda_i x_i \ / \ \lambda_i \geq 0, \sum_{i=1}^{r+1} \lambda_i = 1, \ x_i \in F \right\} \tag{30}$$

Then let $x \in \overline{\mathrm{Co}\,F}$, i.e.:

$$x = \lim_{k \rightarrow \infty} x^k = \lim_{k \rightarrow \infty} \sum_{i=1}^{r+1} \lambda_i^k \, x_i^k$$

with

$$x^k = \sum_{i=1}^{r+1} \lambda_i^k \, x_i^k \ , \qquad \sum_{i=1}^{r+1} \lambda_i^k = 1 \ , \quad \lambda_i^k \geq 0 \ , \ x_i^k \in F$$

But $\left\{ \lambda_i^k \right\}_k \subset [0,1]$ then, there exists a convergent subsequence (that we still denote λ_i^k) such that:

$$\lambda_i^k \rightarrow \lambda_i \ \text{ if } \ k \rightarrow \infty \ , \ \ \forall i=1,\ldots,r+1 \ , \ \text{ with } \lambda_i \in [0,1] \ , \ \sum_{i=1}^{r+1} \lambda_i = 1$$

Analogously we have $x_i^k \rightarrow x_i \in F$, as $k \rightarrow \infty$, $\forall i$ because $\left\{ x_i^k \right\}_k \subset F$ and F is compact. By virtue of this observation and (30) we have:

$$x = \lim_{k \to \infty} x^k = \sum_{i=1}^{r+1} \lambda_i \, x_i \in \mathrm{CoF},$$

Therefore, Co F is closed, i.e. $\mathrm{CoF} = \overline{\mathrm{CoF}}$.

Finally we will prove that $w_0 \in$ Co F. By definition of integral, w_0 is limit of integrals of step functions defined in a partition $\{I_i\}$ i=1,...,k (with I_i measurable) of the interval [0,1], i.e.

$$w_0 = \int_0^1 f(\alpha(t)) \, dt = \lim_{k \to \infty} \sum_{i=1}^{k} \int_0^1 f(\alpha_i^k) \, \chi_i^k(t) \, dt = \lim_{k \to \infty} \sum_{i=1}^{k} f(\alpha_i^k) \, m_i^k,$$

where $\chi_i^k(t) = \left| \begin{array}{ll} 1 & \text{if } t \in I_i \\ & \\ 0 & \text{in other case} \end{array} \right.$ and $\displaystyle\sum_{i=1}^{k} m_i^k = 1,$

then, by virtue of (30)

$$\sum_{i=1}^{k} f(\alpha_i^k) \, m_i^k = x^k \in \mathrm{CoF}, \; \forall k,$$

and consequently $w_0 = \lim_{k \to \infty} x^k \in$ Co F because CoF is closed.

$\square$

4.2 Approximation of Controls with Step-Functions

• By virtue of proposition 4.1 and (30) we have:

$$w_0 = \int_0^1 f(\alpha(t))\, dt = \sum_{i=1}^{r+1} \lambda_i f(\alpha_i)$$

with

$$\sum_{i=1}^{r+1} \lambda_i = 1 , \quad \lambda_i \geq 0 , \quad f(\alpha_i) \in F , \quad \alpha_i \in A.$$

Then, there exists a partition $\{I_i\}$, $i=1,\ldots,r+1$, of interval $[0,1]$, with measure of $I_i = \lambda_i$ and a step function $\alpha_W(t)$, with $\alpha_W(t)=\alpha_i \in A$, if $t \in I_i$, such that:

$$w_0 = \int_0^1 f(\alpha_W(t))\, dt = \int_0^1 f(\alpha(t))\, dt$$

By a simple argument, we obtain the same result for an arbitrary interval $[a,b]$, i.e. there exists a step function with at most $r+1$ steps such that:

$$w_0 = \int_a^b f(\alpha(t))\, dt = \int_a^b f(\alpha_W(t))\, dt \tag{31}$$

Finally, by virtue of (31), the following lemma is self-evident.

Lemma 4.1: *Let* $b: \Re^{m+1} \rightarrow \Re^r$ *be a bounded continuous function,* $t_i = \frac{iT}{\mu}$, $i=0,1\ldots,\mu$-1 *a discretization of interval* $[0,T]$, *then there exists a step policy* α_W *with values in* A *such that:*

a) α_W *has at most* $r+1$ *steps in each interval* $[t_i, t_{i+1}]$

b) $$\int_{t_i}^{t_{i+1}} b(t_i,\alpha(s))\, ds = \int_{t_i}^{t_{i+1}} b(t_i,\alpha_W(s))\, ds$$

Remark 4.1: From here, we will denote $Q(\mu,T,b(.,.),\alpha(.))$ the mapping with the above property, i.e.:

$$\alpha_W(.) = Q(\mu,T,b(.,.),\alpha(.)) \quad \text{implies} \quad \int_{t_i}^{t_{i+1}} b(t_i,\alpha(s))\, ds = \int_{t_i}^{t_{i+1}} b(t_i,\alpha_W(s))\, ds \quad i=0,\ldots,\mu-1$$

The following lemma establishes an estimate for the difference between the original trajectory of the system and the response corresponding to the approximating step control function α_W.

Lemma 4.2: *Let* $y(\cdot)$ *be the response corresponding to the control* $\alpha(\cdot)$ (*i.e. the solution of* (9)). *Let us consider a partition of interval* $[0,T]$ *in* n_1 *intervals of length* $h_1 = T/n_1$ *and*

$$y_w(t) = x + \int_0^t g(y_w(s),\alpha_w(s))\, ds \quad \forall\ t \in [0,T],$$

where $\alpha_w = Q(\mu, T, b(.,.), \alpha(.))$ *and the function* $b: \Re^{m+1} \rightarrow \Re^{\nu+1}$ *is defined by:*

$$b(t, \alpha) = (g'(y(t),\alpha), f(y(t),\alpha))'$$

then

$$\| y(t) - y_w(t) \| < 3\, M_g\, h_1\, e^{L_g t}$$

Proof: Let be

$$E_i = \|y(t_i) - y_w(t_i)\| = \left\| \int_0^{t_i} g(y(s),\alpha(s))\, ds - \int_0^{t_i} g(y_w(s),\alpha_w(s))\, ds \right\|$$

then

$$E_{i+1} \le E_i + \left\| \int_{t_i}^{t_{i+1}} \Big(g(y(s),\alpha(s)) - g(y_w(s),\alpha_w(s)) \Big)\, ds \right\| \le$$

$$\le E_i + \int_{t_i}^{t_{i+1}} \|g(y(s),\alpha(s)) - g(y(t_i),\alpha(s))\|\, ds + \left\| \int_{t_i}^{t_{i+1}} \Big(g(y(t_i),\alpha(s)) - g(y(t_i),\alpha_w(s)) \Big)\, ds \right\| +$$

$$+ \int_{t_i}^{t_{i+1}} \|g(y(t_i),\alpha_w(s)) - g(y_w(s),\alpha_w(s))\|\, ds \tag{32}$$

By virtue of lemma 4-1 the third term at the right side of inequality (32) is zero, i.e.:

$$\left\| \int_{t_i}^{t_{i+1}} \Big(g(y(t_i),\alpha(s)) - g(y(t_i),\alpha_w(s)) \Big)\, ds \right\| = 0$$

Applying inequality (10) and considering that:

$$\| y(s) - y(t_i) \| \le M_g\, (s - t_i) \quad \forall\ s \in [t_i, t_{i+1}),$$

we have:

$$E_{i+1} \le E_i + \int_{t_i}^{t_{i+1}} L_g\, M_g\, (s - t_i)\, ds + \int_{t_i}^{t_{i+1}} L_g\, \| y(t_i) - y_w(s) \|\, ds$$

finally, taking in mind that:

$$\|y(t_i) - y_w(s)\| \leq \|y(t_i) - y_w(t_i)\| + \|y_w(t_i) - y_w(s)\|$$

the previous inequality becomes:

$$E_{i+1} \leq E_i + \int_{t_i}^{t_{i+1}} L_g M_g (s-t_i)\, ds + \int_{t_i}^{t_{i+1}} L_g(E_i + M_g(s-t_i)\, ds \leq$$

$$\leq E_i + L_g M_g \frac{h_1^2}{2} + L_g E_i h_1 + L_g M_g \frac{h_1^2}{2} = E_i\,(1+L_g h_1) + L_g M_g\, h_1^2 \tag{33}$$

From (33), it follows by induction:

$$E_i \leq L_g M_g h_1^2 \frac{(1+L_g h_1)^{i-1}}{1+L_g h_1 - 1} = M_g\, h_1 (1+L_g h_1)^{i-1} \tag{34}$$

Let be $t \in [t_i, t_{i+1}]$, $0 \leq i \leq n_1$, then

$$\| y(t) - y_w(t) \| = \left\| \int_0^t g(y(s),\alpha(s))\, ds - \int_0^t g(y_w(s),\alpha_w(s))\, ds \right\| \leq$$

$$\leq E_i + \left\| \int_{t_i}^t g(y(s),\alpha(s))\, ds - \int_{t_i}^t g(y_w(s),\alpha_w(s))\, ds \right\| \leq$$

$$\leq E_i + \int_{t_i}^t \|g(y(s),\alpha(s)) - g(y_w(s),\alpha_w(s))\|\, ds \leq E_i + 2M_g(t-t_i)$$

By virtue of (34), this last inequality becomes:

$$\| y(t)-y_w(t)\| \leq M_g\, h_1 (1+L_g \tfrac{T}{n_1})^{i-1} + 2M_g h_1 \leq M_g h_1 \left(e^{L_g t_i} + 2e^{L_g t_i}\right) \leq 3M_g h_1 e^{L_g t} \tag{35}$$

In consequence

$$\| y(t) - y_w(t)\| \leq 3\, M_g\, h_1\, e^{L_g t}$$

□

4.3 Approximation of Controls with Uniform-Step Functions

The control policy given by lemma 4.2 is a step policy and it has at most $\nu+2$ steps in each interval of length $h_1=\frac{T}{n_1}$. To relate this type of functions with those belonging to $\mathcal{A}^h$ and $\mathcal{A}^h_T$, it is necessary to approximate them with step functions where the step has a constant length h_2. We introduce the parameter n_2 and we define:

$$h_2=\frac{T}{n_1\,n_2(\nu+2)}=\frac{h_1}{n_2(\nu+2)} \tag{36}$$

We assign to each function α_W a function α^h_w (with steps of equal length h_2) by the following procedure:

For each interval $[t_i,\ t_{i+1})$, $t_i=(i-1)h_1$, we know that α_W takes at most $(\nu+2)$ different values that we denote α_{ij}, $i=1,\ n_1$; $j=0,\ \dots\ \nu+1$, we will denote λ_{ij} the length of the subinterval where α_W takes the value α_{ij} and

$$n_{ij}=[\lambda_{ij}\ /\ h_2] \tag{37}$$

where [s] denote the integer part of a real number s.
We define the new function α^h_w in such a way that it coincides with α_W, at least in a subinterval of length $n_{ij}h_2$, where it takes the value α_{ij}, i.e.:

$$\alpha^h_w(s)=\alpha_{ij}\ \forall\ s\in[\hat{t}_{ij},\ \hat{t}_{i,j+1}) \tag{38}$$

where:

$$t_{i0}=t_i\quad\text{for}\ \ j=0$$

$$t_{ij}=t_i+\sum_{r=0}^{j-1}\lambda_{ij}\quad\text{for}\ j=1,\ \nu+2 \tag{39}$$

$$\hat{t}_{ij}=h_2\,[t_{ij}/h_2]\quad\text{for}\ j=0,\ \nu+2$$

Then α^h_w coincides with α_W in $[t_i,\ t_{i+1})$ except in at most $(\nu+1)$ intervals of residual length $\tilde{\eta}_{ij}$, with:

$$\tilde{\eta}_{ij}=t_{ij}-\hat{t}_{ij}<h_2$$

and so in a set of measure $\tilde{\eta}_i$ such that:

$$\tilde{\eta}_i < (\nu+1)h_2 \tag{40}$$

With this considerations we can prove lemma 4.3. This lemma establishes an estimate for the difference between the original trajectory of the system and the response corresponding to the approximating control function with uniform steps α_w^h.

Lemma 4.3: *Let* $y(\cdot)$ *be the response corresponding to the control* $\alpha(\cdot)$ *(i.e. the solution of* (9)*). Let us consider a partition of interval* $[0,T]$ *in* μ *intervals of length* h, $\mu=(\nu+2)n_1 n_2$ *and*

$$y_w^h(t) = x + \int_0^t g(y_w^h(s),\alpha_w^h(s))\, ds \ \forall\ t \in [0,T]$$

where α_w^h *is obtained applying the above construction, then:*

$$\left\| y(t) - y_w^h(t) \right\| < \bar{K}\sqrt{h}\ e^{L_g t} \tag{41}$$

Proof: Let be

$$E_i = \left\| \int_0^{t_i} g(y(s),\alpha(s))\, ds - \int_0^{t_i} g(y_w^h(s),\alpha_w^h(s))\, ds \right\| = \left\| y(t_i) - y_w^h(t_i) \right\| \quad i=1,\dots,n_1 \tag{42}$$

then

$$E_{i+1} \le E_i + \left\| \int_{t_i}^{t_{i+1}} \Big(g(y(s),\alpha(s)) - g(y_w^h(s),\alpha_w^h(s)) \Big)\, ds \right\| \le$$

$$\le E_i + \int_{t_i}^{t_{i+1}} \left\| g(y(s),\alpha(s)) - g(y(t_i),\alpha(s)) \right\| ds + \left\| \int_{t_i}^{t_{i+1}} \Big(g(y(t_i),\alpha(s)) - g(y(t_i),\alpha_w(s)) \Big)\, ds \right\| +$$

$$+ \int_{t_i}^{t_{i+1}} \left\| g(y(t_i),\alpha_w(s)) - g(y(t_i),\alpha_w^h(s)) \right\| ds + \int_{t_i}^{t_{i+1}} \left\| g(y(t_i),\alpha_w^h(s)) - g(y_w^h(s),\alpha_w^h(s)) \right\| ds \tag{43}$$

By virtue of lemma 4-1 the third term of (43) is zero, i.e.:

$$\left\| \int_{t_i}^{t_{i+1}} \Big(g(y(t_i),\alpha(s)) - g(y(t_i),\alpha_w(s)) \Big)\, ds \right\| = 0,$$

The fourth term of (43) can be bounded, by virtue of (40) in the following way:

$$\int_{t_i}^{t_{i+1}} \left\| g(y(t_i),\alpha_w(s)) - g(y(t_i),\alpha_w^h(s)) \right\| ds \leq 2\, \tilde{\eta}_i\, M_g \tag{44}$$

because

$$g(y(t_i),\alpha_w(s)) = g(y(t_i),\alpha_w^h(s))$$

except in sub-intervals of total length $\tilde{\eta}_i$.

Applying inequality (10) and considering that:

$$\| y(s) - y(t_i) \| \leq M_g\, (s - t_i) \quad \forall\, s \in [t_i, t_{i+1}),$$

we have::

$$E_{i+1} \leq E_i + \int_{t_i}^{t_{i+1}} L_g M_g (s-t_i)\, ds + \int_{t_i}^{t_{i+1}} L_g \left\| y(t_i) - y_w^h(s) \right\| ds + 2\, M_g\, (\nu+1) h$$

and having in mind that:

$$\left\| y(t_i) - y_w^h(s) \right\| \leq \left\| y(t_i) - y_w^h(t_i) \right\| + \left\| y_w^h(t_i) - y_w^h(s) \right\|$$

the last inequality becomes:

$$E_{i+1} \leq E_i + \int_{t_i}^{t_{i+1}} L_g M_g (s-t_i)\, ds + \int_{t_i}^{t_{i+1}} L_g (E_i + M_g (s-t_i)\, ds + 2\, M_g\, (\nu+1) h \leq$$

$$\leq E_i + L_g M_g \frac{h_1^{\,2}}{2} + L_g E_i h_1 + L_g M_g \frac{h_1^{\,2}}{2} + 2\, M_g\, (\nu+1) h =$$

$$= E_i\, (1 + L_g h_1) + L_g M_g h_1^{\,2} + 2\, M_g\, (\nu+1) h \tag{45}$$

From (45) it follows easily:

$$E_i \leq \left(L_g M_g h_1^{\,2} + 2\, M_g\, (\nu+1) h \right) \frac{(1 + L_g h_1)^{i-1}}{1 + L_g h_1 - 1} =$$

$$= \left(M_g h_1 + \frac{2\, M_g\, (\nu+1) h}{L_g h_1} \right) (1 + L_g h_1)^{i-1} = \left(M_g n_2 (\nu+1) h + \frac{2\, M_g}{L_g\, n_2} \right) (1 + L_g h_1)^{i-1}$$

Taking $n_2 = [(hL_g(\nu+1)/2)^{-1/2}]$, we obtain:

$$E_i \leq K\, h^{1/2}\, (1+L_g h_1)^{i-1} \tag{46}$$

Let be $t \in [t_i, t_{i+1}]$, $0 \leq i \leq n$, then:

$$\left\| y(t) - y_w^h(t) \right\| = \left\| \int_0^t g(y(s),\alpha(s))\, ds - \int_0^t g(y_w^h(s),\alpha_w^h(s))\, ds \right\| \leq$$

$$\leq E_i + \left\| \int_{t_i}^t g(y(s),\alpha(s))\, ds - \int_{t_i}^t g(y_w^h(s),\alpha_w^h w(s))\, ds \right\| \leq$$

$$\leq E_i + \int_{t_i}^t \left\| g(y(s),\alpha(s)) - g(y_w^h(s),\alpha_w^h w(s)) \right\| ds \leq$$

$$\leq E_i + 2M_g(t-t_i)$$

By virtue of (46), this last inequality becomes:

$$\left\| y(t)-y_w^h(t) \right\| \leq K\, h^{1/2} (1+L_g \tfrac{T}{n})^{i-1} + 2M_g h_1 \leq \left(K\, h^{1/2} + 2M_g h_1 \right) e^{L_g t_i}$$

In consequence:

$$\left\| y(t) - y_w^h(t) \right\| \leq \bar{K}\, e^{L_g t}\, h^{1/2}$$

□

5. RATE OF CONVERGENCE

5.1 Convergence in the Case with Finite Horizon:

Theorem 5.1: *Let us consider a partition of interval* $[0,T]$ *in* $n=(\nu+2)\,n_1\,n_2$ *intervals of length* $h=T/n$, *then*

$$\left| u_T(0,x) - u_T^h(0,x) \right| \leq \mathcal{C}\,\phi(T)\,h^{1/2} \tag{47}$$

where $\phi(T)$ *is defined by*

$$\phi(T) = \begin{cases} e^{(L_g-\lambda)T} & \text{if } L_g > \lambda \\ T & \text{if } L_g = \lambda \\ 1 & \text{if } L_g < \lambda \end{cases} \tag{48}$$

Proof: let be

$$u_T^e(t,x) = \min_{\alpha \in \mathcal{A}_T^h} J_T(t,x,\alpha(\cdot)) \tag{49}$$

then

$$\left| u_T(0,x) - u_T^h(0,x) \right| \leq \left| u_T(0,x) - u_T^e(0,x) \right| + \left| u_T^e(0,x) - u_T^h(0,x) \right| \tag{50}$$

It is easy to prove (see [2]) that:

$$\left| u_T^e(0,x) - u_T^h(0,x) \right| \leq C\,\phi(T)\,h \tag{51}$$

The meaning of (51) is simply the bound of the error associated to the Euler's method of integration of (9). The proof of (51) is contained in [2].

It remains now to obtain an estimate for $\left| u_T(0,x) - u_T^e(0,x) \right|$

Let $\alpha(\cdot)$ be an arbitrary control, if we apply to it the process of approximation with uniform length step functions described in §4.3 , we have:

$$\left| \int_0^T \Big(f(y(s),\alpha(s)) - f(y_w^h(s),\alpha_w^h(s)) \Big) e^{-\lambda s} ds \right| \le$$

$$\le \int_0^T | f(y(s),\alpha(s)) - f(y_e(s),\alpha(s)) | e^{-\lambda s} ds + \left| \int_0^T \Big(f(y_e(s),\alpha(s)) - f(y_e(s),\alpha_w(s)) \Big) e^{-\lambda s} ds \right| +$$

$$+ \int_0^T | f(y_e(s),\alpha_w(s)) - f(y(s),\alpha_w(s)) | e^{-\lambda s} ds + \int_0^T | f(y(s),\alpha_w(s)) - f(y(s),\alpha_w^h(s)) | e^{-\lambda s} ds$$

$$+ \int_0^T | f(y(s),\alpha_w^h(s)) - f(y_w^h(s),\alpha_w^h(s)) | e^{-\lambda s} ds \tag{52}$$

where

$$y_e(s) = y(t_i) \text{ if } s \in [t_i, t_{i+1}),\ i=1,\ldots,n-1.$$

We are going to study each terms of (52), taking in mind in particular that:

$$\| y(s) - y_e(s) \| \le M_g\,(s - t_i) \quad \forall\, s \in [t_i, t_{i+1}) \tag{53}$$

By virtue of (11) and (53) the first term of (52) becomes:

$$\int_0^T \left| \Big(f(y(s),\alpha(s)) - f(y_e(s),\alpha(s)) \Big) \right| e^{-\lambda s} ds \le \int_0^T L_f \| y(s) - y_e(s) \| e^{-\lambda s} ds \le$$

$$\le M_g\, L_f \sum_{i=0}^{n_1-1} \int_{t_i}^{t_{i+1}} (s - t_i)\, e^{-\lambda s}\, ds \le$$

$$\le M_g\, L_f\, h \sum_{i=0}^{n_1-1} \int_{t_i}^{t_{i+1}} e^{-\lambda s} ds \le M_g\, L_f\, \frac{h}{\lambda}\, (1 - e^{-\lambda T}) \tag{54}$$

In the same way we can estimate the third term of (52), i.e.

$$\int_0^T | f(y_e(s),\alpha_w(s)) - f(y(s),\alpha_w(s)) | e^{-\lambda s} ds \le M_g\, L_f\, \frac{h}{\lambda}\, (1 - e^{-\lambda T}).$$

We can estimate the second term of (52) in the following way:

$$\left|\int_0^T\Big(f(y_e(s),\alpha(s))-f(y_e(s),\alpha_w(s))\Big)e^{-\lambda s}\,ds\right|=\left|\sum_{i=0}^{n_1-1}\int_{t_i}^{t_{i+1}}\Big(f(y(t_i),\alpha(s))-f(y(t_i),\alpha_w(s))\Big)e^{-\lambda s}\,ds\right|\leq$$

$$\leq\left|\sum_{i=0}^{n_1-1}\int_{t_i}^{t_{i+1}}\Big(f(y(t_i),\alpha(s))-f(y(t_i),\alpha_w(s))\Big)e^{-\lambda t_i}\,ds\right|+2M_f\sum_{i=0}^{n_1-1}\int_{t_i}^{t_{i+1}}\left|e^{-\lambda s}-e^{-\lambda t_i}\right|ds\leq$$

$$\leq 2\,M_f\,(1-e^{-\lambda T})\,h \tag{55}$$

because, by lemma 4.1:

$$\left|\sum_{i=0}^{n}\int_{t_i}^{t_{i+1}}\Big(f(y(t_i),\alpha(s))-f(y(t_i),\alpha_w(s))\Big)e^{-\lambda t_i}\,ds\right|=0$$

The fourth term of (52) can be bounded, by virtue of (40), in this way:

$$\int_0^T\left|f(y(s),\alpha_w(s))-f(y(s),\alpha_w^h(s))\right|e^{-\lambda s}ds\leq 2M_f\sum_{i=0}^{n_1-1}e^{-\lambda h_1 i}\,(\nu+1)h\leq$$

$$\leq 2\,M_f\,(1-e^{-\lambda h_1})^{-1}\,(\nu+1)\,h$$

Finally, taking into account (36) and that $n_2=[(hL_g(\nu+1)/2)^{-1/2}]$, the last expression becomes:

$$\int_0^T\left|f(y(s),\alpha_w(s))-f(y(s),\alpha_w^h(s))\right|e^{-\lambda s}ds\leq 2\,M_f\,(1-e^{-\lambda h_1})^{-1}\,(\nu+1)\,h\leq M\sqrt{h} \tag{56}$$

By virtue of (11) and lemma 4-2 the last term of (52) can be bounded in the following way, if $L_g\neq\lambda$:

$$\int_0^T\left|f(y(s),\alpha_w^h(s))-f(y_w^h(s),\alpha_w^h(s))\right|e^{-\lambda s}ds\leq\int_0^T L_f\left\|y(s)-y_w^h(s)\right\|e^{-\lambda s}\,ds\leq$$

$$\leq\bar{K}\left(\frac{e^{(L_g-\lambda)T}-1}{L_g-\lambda}\right)\sqrt{h} \tag{57}$$

If $L_g=\lambda$, we would obtain

$$\int_0^T\left|f(y(s),\alpha_w^h(s))-f(y_w^h(s),\alpha_w^h(s))\right|e^{-\lambda s}ds\leq\bar{K}\,L_f\,T\sqrt{h} \tag{58}$$

In consequence, inequality (52) becomes, in the case $L_g\neq\lambda$:

$$\left|\int_0^T\Big(f(y(s),\alpha(s))-f(y_w^h(s),\alpha_w^h(s))\Big)e^{-\lambda s}ds\right| \le 2\ M_g\ L_f\ \frac{h}{\lambda}\ (1-e^{-\lambda T})\ +\ 2\ M_f\ (1-e^{-\lambda T})h\ +$$

$$+\ M\sqrt{h}\ +\ \bar{K}\ \Big(\frac{e^{(L_g-\lambda)T}-1}{L_g-\lambda}\Big)\sqrt{h} \qquad (59)$$

and in the case $L_g=\lambda$

$$\left|\int_0^T\Big(f(y(s),\alpha(s))-f(y_w^h(s),\alpha_w^h(s))\Big)e^{-\lambda s}ds\right| \le 2\ M_g\ L_f\ \frac{h}{\lambda}\ (1-e^{-\lambda T})\ +\ 2\ M_f\ (1-e^{-\lambda T})h\ +$$

$$+\ M\sqrt{h}\ +\ \bar{K}\ L_f\,T\sqrt{h} \qquad (60)$$

Using a compact notation, by virtue of (59) and (60) we get the following inequality:

$$\forall\alpha(\cdot)\ \exists\ \alpha_w^h(\cdot)\ /\ \left|\int_0^T\Big(f(y(s),\alpha(s))-f(y_w^h(s),\alpha_w^h(s))\Big)e^{-\lambda s}ds\right| \le\ M\ \phi(T)\sqrt{h}\ ,$$

therefore

$$u_T^e(0,x)\ =\ \min_{\alpha\in\mathcal{A}^h}\ J_T(0,x,\alpha)\ \le\ \inf_{\alpha\in\mathcal{A}}\ J_T(0,x,\alpha)\ +\ M\ \phi(T)\sqrt{h}\ =\ u_T(0,x)\ +\ M\ \phi(T)\sqrt{h}$$

On the other hand, obviously $u_T(0,x)\le u_T^e(0,x)$; then: $\forall T$

$$\left|u_T^e(0,x)\ -\ u_T(0,x)\right| \le M\ \phi(T)\sqrt{h} \qquad (61)$$

In this way, from (51) and (61) we obtain

$$\left|\ u_T(0,\ x)\ -\ u_T^h(0,\ x)\ \right| \le\ \bar{M}\ \phi(T)\sqrt{h}$$

□

5.2 Convergence in the Case with Infinite Horizon .

The procedure to prove the convergence in this case, is fundamentally based in the application of the result of convergence for the case of finite horizon.

Theorem 5.2:

$$\left\| u(x) - u^h(x) \right\| \leq \mathcal{C}\, h^{\gamma/2} \tag{62}$$

where

$$\gamma=1 \qquad \textit{if } \lambda > L_g$$

$$\gamma=\frac{\lambda}{L_g} \qquad \textit{if } \lambda < L_g,$$

$$\gamma\in(0,1) \qquad \textit{if } \lambda = L_g.$$

Proof:

$$\left|u(x) - u^h(x)\right| \leq \left|u(x) - u_T(x)\right| + \left|u_T(x) - u_T^h(x)\right| + \left|u_T^h(x) - u^h(x)\right| \tag{63}$$

It is easy to see that:

$$\left| u(x) - u_T(x) \right| \leq 2\,\frac{M_f}{\lambda}\, e^{-\lambda T} \tag{64}$$

By Theorem 5.1 we have:

$$\left| u_T(x) - u_T^h(x) \right| \leq \bar{M}\, \phi(T)\, \sqrt{h} \tag{65}$$

Finally, as:

$$\left| u_T^h(x) - u^h(x) \right| \leq 2\, M_f\, e^{-\lambda T} \tag{66}$$

we obtain

$$\left| u(x) - u^h(x) \right| \leq \bar{M}\, \phi(T)\, \sqrt{h} + 4\, M_f\, e^{-\lambda T} \tag{67}$$

To obtain (62), we will deal with the following cases:

- Case $L_g > \lambda$:

By (67) we have

$$\left| u(x) - u^h(x) \right| \leq M_1 \left(e^{-\lambda T} + \sqrt{h}\, e^{(L_g-\lambda)T} \right) \tag{68}$$

The expression (68) has a minimum in T_1, given by:

$$T_1 = \frac{1}{L_g} \ln\left(\frac{\lambda}{L_g - \lambda}\, h^{-1/2} \right)$$

in consequence defining $\gamma = \frac{\lambda}{L_g}$

$$e^{(L_g-\lambda)T} = \left(\frac{\lambda}{L_g-\lambda} \right)^{1-\gamma} h^{(\gamma-1)/2}$$

$$e^{-\lambda T} = \left(\frac{L_g-\lambda}{\lambda} \right)^{\gamma} h^{\gamma/2}$$

Finally, replacing in (68) we have:

$$\left| u^h(x) - u(x) \right| \leq M_1 K_1\, h^{1/2}\, h^{(\gamma-1)/2} + M_1 K_1\, h^{\gamma/2} = 2 M_1 K_1 h^{\gamma/2} \tag{69}$$

where

$$K_1 = \max \left\{ \left(\frac{\lambda}{L_g-\lambda} \right)^{1-\gamma}, \left(\frac{L_g-\lambda}{\lambda} \right)^{\gamma} \right\}$$

By virtue of (69) we have:

$$\left| u(x) - u^h(x) \right| \leq \mathcal{C}\, h^{\gamma/2} \qquad \text{if } L_g > \lambda$$

• Case $L_g < \lambda$:

In this case we have that (67) becomes

$$\leq 4 M_f\, e^{-\lambda T} + \bar{M} \sqrt{h}$$

and taking $T \to \infty$ we obtain:

$$\left| u(x) - u^h(x) \right| \leq \bar{M} \sqrt{h} \tag{70}$$

• Case $L_g = \lambda$:

By (67)

$$\left| u(x) - u^h(x) \right| \leq \bar{M}\, T \sqrt{h} + 4 M_f e^{-\lambda T} \leq M_2 \left(e^{-\lambda T} + \sqrt{h}\, T\right) \tag{71}$$

The minimum of expression (71) is realized by:

$$T = -\frac{1}{\lambda} \ln\frac{\sqrt{h}}{\lambda}, \text{ if } \sqrt{h} < \lambda,$$

then expression (71) becomes:

$$\left| u(x) - u^h(x) \right| \leq -M_2 \frac{\sqrt{h}}{\lambda} \ln\frac{\sqrt{h}}{\lambda} + M_2 \frac{\sqrt{h}}{\lambda} \tag{72}$$

now we prove that $\forall\ \gamma \in (0,1)$ there exists $K > 0$ such that:

$$- x \ln x \leq K x^{\gamma} \tag{73}$$

In effect:

$$- x \ln x \leq K_2 x^{\gamma} \Leftrightarrow - \ln x \leq K_2 x^{-q}, \Leftrightarrow x^q \ln \frac{1}{x} \leq K_2, \text{ with } q = 1-\gamma.$$

Let be

$$t(x) = x^q \ln \frac{1}{x} \geq 0,\ x \in (0,1)$$

As $t(1)=0$ and $\lim_{x\to 0} t(x)=0$, then $t(x)$ has a maximum at $\hat{x} = e^{-\frac{1}{q}}$, so

$$t(x) \leq t(e^{-\frac{1}{q}}) = \frac{1}{q} e^{-1} = K_2$$

By virtue of (72) and (73) we have

$$\left\| u(x) - u^h(x) \right\| \leq \mathcal{C}\, h^{\gamma/2} \qquad \text{if } L_g = \lambda, \text{ with } \gamma \in (0,1)$$

where $\mathcal{C}$ depends on γ and $\mathcal{C} \to \infty$ when $\gamma \to 1$ □

5.3 Convergence in the Case of Semiconcave Functions

In the case that functions f, g have more regularity properties, it is possible to prove that also the optimal cost function u is more regular and to use that fact to prove a better estimate for the error $\| u - u^h \|$ in the set $\lambda > L_g$.

In addition to the convex analysis results used throughout this paper, other key points to get this result is the use of the family of functions $u_T^h(n,x)$ and the proof of the semiconcavity property of them. By virtue of it, the regular approximation $u_{\rho,T}^h$ has a hessian matrix uniformly bounded from above. This result enables us to obtain a suitable estimate for $\| u_T - u_T^h \|$ using simple techniques of interpolation theory of differentiable functions. In terms of it, the corresponding estimate of the error $\| u - u^h \|$ follows easily employing the same procedure used in theorem 5.2.

We assume here that f, g have the following semiconcavity property, for any $\alpha \in A$

$$H_1 \left| \begin{array}{ll} \| g(x+z,\alpha) - 2\, g(x,\alpha) + g(x-z,\alpha) \| \le C \| z \|^2 & (74) \\ \\ | f(x+z,\alpha) - 2\, f(x,\alpha) + f(x-z,\alpha) | \le C \| z \|^2 & (75) \end{array} \right.$$

Theorem 5.3: *Under assumptions* H_1, *we have that* $\forall\, n / 0 \le n \le \mu$, $u_T^h(n,\cdot)$ *has the following semiconcavity property*

$$u_T^h(n,x+z) - 2\, u_T^h(n,x) + u_T^h(n,x-z) \le \psi(T-nh) \| z \|^2 \tag{76}$$

where

$$\psi(t) = \left| \begin{array}{ll} M\, e^{(2L_g - \lambda)t} & if\ 2L_g > \lambda \\ M\, t & if\ 2L_g = \lambda \\ M & if\ 2L_g < \lambda \end{array} \right. \tag{77}$$

Proof: By (23) and (24)

$$u_T^h(n,x+z) - 2\, u_T^h(n,x) + u_T^h(n,x-z) \le \sup_{\alpha \in \mathcal{A}_T^h} \Big(J_T^h(n,\, x+z,\, \alpha) - 2\, J_T^h(n,\, x,\, \alpha) + J_T^h(n,\, x-z,\, \alpha) \Big)$$

then, $\forall\ \alpha \in \mathcal{A}_T^h$, we have

$$u_T^h(n,x+z) - 2\, u_T^h(n,x) + u_T^h(n,x-z) \le h \sum_{k=n}^{\mu-1} A_k \left(1 - \lambda h\right)^{k-n} \tag{78}$$

where $\alpha_k = \alpha(kh)$, and

$$A_k = f\Big(y_h(x,k)+\big(y_h(x+z,k)-y_h(x,k)\big),\alpha_k\Big) - 2f\Big(y_h(x,k),\alpha_k\Big) + f\Big(y_h(x,k)-\big(y_h(x+z,k)-y_h(x,k)\big),\alpha_k\Big) +$$

$$+ f\Big(y_h(x-z,k),\alpha_k)\Big) - f\Big(y_h(x,k)-\big(y_h(x+z,k)-y_h(x,k)\big),\alpha_k\Big)$$

By (11) and (75) we can estimate A_k in the following way

$$A_k \leq C \| y_h(x+z,k) - y_h(x,k) \|^2 + L_f \| y_h(x+z,k) - 2y_h(x,k) + y_h(x-z,k) \| \tag{79}$$

By virtue of (10) and (25) it follows

$$\| y_h(x+z,k) - y_h(x,k) \| \leq (1+L_g h)^k \| z \| \tag{80}$$

Taking into account (80) and using an argument similar to those used above to obtain the estimate (79), we can also obtain the following estimate:

$$\| y_h(x+z,k)-2y_h(x,k)+ y_h(x-z,k) \| \leq (1 + L_g h) \| y_h(x+z,k\text{-}1)-2y_h(x,k\text{-}1)+ y_h(x-z,k\text{-}1)\| +$$

$$+ C h\, (1+L_g h)^{2k} \| z \|^2$$

From here it follows, by a simple induction procedure

$$\| y_h(x+z,k) - 2y_h(x,k) + y_h(x-z,k) \| \leq M h\, (1 + L_g h)^{k-1} \frac{(1 + L_g h)^k}{1-(1 + L_g h)} \| z \|^2$$

Finally, replacing in (79) we obtain

$$A_k \leq M(1+L_g h)^{2k} \| z \|^2$$

Now, replacing in (78) we have

$$u_T^h(n,x+z) - 2u_T^h(n,x) + u_T^h(n,x-z) \leq M h \| z \|^2 \sum_{k=n}^{\mu-1} (1+L_g h)^{2(k-n)} \big(1-\lambda h\big)^{k-n} \leq$$

$$\leq M h \| z \|^2 \sum_{k=n}^{\mu-1} e^{(2L_g-\lambda)h(k-n)} \leq M h \| z \|^2 \psi(T-nh)$$

□

5.3.1 *Definition of* $u^h_{\rho,T}$, *regularization of* u^h_T

By convolution with a smooth function $\beta(\cdot)$, we obtain a regular approximation of u^h_T

$$u^h_{\rho,T}(n,x) = \left(u^h_T * \beta_\rho\right)(x) = \int_{B(\rho)} u^h_T(n,x-y)\, \beta_\rho(y)\, dy \tag{81}$$

where

$\beta_1(\cdot) \in C^\infty(\Re^\nu)$,

$\beta_1(x) \geq 0 \ \forall x$, support of $\beta_1 \subset B_1 = \{ x \in \Re^\nu \ / \ \| x \| \leq 1 \}$

$\int_{\Re^\nu} \beta_1(x)\, dx = 1$

$\beta_\rho(x) = \frac{1}{\rho^\nu}\, \beta_1(\frac{x}{\rho}) \geq 0 \quad \forall \rho \in \Re^+$

It is easy to prove that $u^h_{\rho,T}$ verifies the properties established in the following propositions:

By convolution of (76), it follows the semiconcavity of $u^h_{\rho,T}$:

Proposition 5.1:

$$u^h_{\rho,T}(n,x+z) - 2\, u^h_{\rho,T}(n,x) + u^h_{\rho,T}(n,x-z) \leq \psi(T-nh)\, \| z \|^2 \tag{82}$$

As $u^h_{\rho,T}(n, \cdot) \in C^\infty$, taking limit in (82), we obtain:

Proposition 5.2: $u^h_{\rho,T}(n, \cdot)$ *has a hessian matrix bounded from above in the following sense:*

$$\left(w, \frac{\partial^2 u^h_{\rho,T}(n,x)}{\partial x^2}\, w \right) \leq \psi(T-nh)\, \| w \|^2 \tag{83}$$

where for $w \in \Re^\nu$, *we consider* (w,v) *the euclidian product and* $\| w \|^2 = (w,w)$.

By definition of convolution, taking into account that $u^h_T(n,x)$ is Lipschitz continuous with constant $L_{u^h_T}$, we have the following estimate of the difference between the function u^h_T and its regularization:

Proposition 5.3:

$$\left|u^h_T(n,x) - u^h_{\rho,T}(n,x)\right| \leq L_{u^h_T}\ \rho \tag{84}$$

By convolution of (21), it holds the following inequality:

Proposition 5.4:

$$u^h_{\rho,T}(n,x) \leq (1-\lambda h)\, u^h_{\rho,T}(n+1,\ x + h g(x,a)\ ) + h f(x,a) + (1-\lambda h) L_{u^h_T} L_g\, h\, \rho \tag{85}$$

Using basic inequalities of interpolation theory, it follows from (83) that:

Proposition 5.5:

$$\sum_{j=0}^{\nu+1} u^h_{\rho,T}(i+1,\ y(t_i) + h g(\ y(t_i), a_{i,j})\ )\ \lambda_{i,j} \leq u^h_{\rho,T}(i+1,\ y(t_i) + h \sum_{j=0}^{\nu+1} g(\ y(t_i), a_{i,j})\ \lambda_{i,j}) + $$
$$+ \ M\ \psi(T-(i+1)h)\, h^2 \tag{86}$$

Where λ_{ij} *verifies*

$$\sum_{j=0}^{\nu+1} \lambda_{ij} = 1\ , \quad \lambda_{ij} \geq 0\ ,$$

and the relation between $\lambda_{i,j}$ *and* $a_{i,j}$ *is given by lemma* 4.1, *i.e.:*

$$\int_{t_i}^{t_{i+1}} f(y(t_i),\ u(s))\ ds = h \sum_{j=0}^{\nu+1} f(y(t_i),\ a_{i,j})\ \lambda_{i,j} \tag{87}$$

The previous properties allow us to obtain the upper bound for $u^h_T(0,x)$ established in the following:

Theorem 5.4: *Under assumptions* (10), (11) *and* H_1 *we have*

$$u_T^h(0,x) \leq u_T(0,x) + \psi(T)\,h \tag{88}$$

Proof: By virtue of (54) and (55) and that $\left| e^{-\lambda t_i} - (1-\lambda h)^i \right| \leq C\,h^2$ we have:

$$\int_0^T f(y(s),\, u(s))\, e^{-\lambda s}\, ds = \sum_{i=0}^{\mu\text{-}1} \left((1-\lambda h)^i \int_{t_i}^{t_{i+1}} f(y(t_i),\, u(s))\, ds \right) + \eta(h)$$

where $|\ \eta(h)\ | \leq M\,h$.

By virtue of lemma 4.1:

$$\int_0^T f(y(s),\, u(s))\, e^{-\lambda s}\, ds = h \sum_{i=0}^{\mu\text{-}1} \left((1-\lambda h)^i \sum_{j=0}^{\nu+1} f(y(t_i),\, a_{i,j})\, \lambda_{i,j} \right) + \eta(h) \tag{89}$$

but by (85)

$$h\, f(y(t_i),\, a_{i,j}) \geq u^h_{\rho,T}(i,\, y(t_i)) - (1-\lambda h)\, u^h_{\rho,T}(i+1,\, y(t_i) + h\, g(y(t_i), a_{i,j})) - (1-\lambda h)\, L_u\, L_g\, h\, \rho$$

then (89) becomes:

$$\int_0^T f(y(s),\, u(s))\, ds \geq$$

$$\geq \sum_{i=0}^{\mu\text{-}1} \left((1-\lambda h)^i \sum_{j=0}^{\nu+1} \left(u^h_{\rho,T}(i,\, y(t_i)) - (1-\lambda h)\, u^h_{\rho,T}(i+1,\, y(t_i) + h\, g(\, y(t_i), a_{i,j})\,) \right) \lambda_{i,j} - \right.$$
$$\left. - \; (1-\lambda h)\, L_u\, L_g\, h\, \rho \right) - \eta(h) =$$

$$= \sum_{i=0}^{\mu\text{-}1} \left((1-\lambda h)^i \left(u^h_{\rho,T}(i,\, y(t_i)) - (1-\lambda h) \sum_{j=0}^{\nu+1} u^h_{\rho,T}(i+1,\, y(t_i) + h\, g(\, y(t_i), a_{i,j})\,)\, \lambda_{i,j} \right) - \right.$$
$$\left. - \; (1-\lambda h)\, L_u\, L_g\, h\, \rho \right) - \eta(h)$$

By (86)

$$\int_0^T f(y(s),\, u(s))\, e^{-\lambda s}\, ds \geq$$

$$\geq \sum_{i=0}^{\mu-1} \left((1-\lambda h)^i \left(u^h_{\rho,T}(i,\, y(t_i)) - (1-\lambda h)\ u^h_{\rho,T}(i+1,\, y(t_i) + h \sum_{j=0}^{\nu+1} g(\, y(t_i), a_{i,j})\ \lambda_{i,j}) + \right.\right.$$

$$\left.\left. +\ M\ \psi(T-(i+1)h)\ h^2 \right) - (1-\lambda h)\, L_u\, L_g\ h\, \rho \right) - \eta(h)$$

as $L_{u^h_{\rho,T}} = L_{u^h_T} = \phi(T-ih)$ and taking into account that

$$y(t_{i+1}) = y(t_i) + h \sum_{j=0}^{\nu+1} g(\, y(t_i), a_{i,j})\ \lambda_{i,j} + \chi^i(h) \tag{90}$$

$$\text{where } \| \chi^i(h) \| \leq M\ h^2$$

we obtain, taking in particular into account that $\phi(t) \leq \psi(t)\ \forall\ t$

$$\int_0^T f(y(s),\, u(s))\, e^{-\lambda s}\, ds \geq \sum_{i=0}^{\mu-1} (1-\lambda h)^i \left(u^h_{\rho,T}(i,\, y(t_i)) - (1-\lambda h)\ u^h_{\rho,T}(i+1,\, y(t_{i+1})) - \right.$$

$$\left. - M \Big(\phi(T-(i+1)h) + \psi(T-(i+1)h) \Big) h^2 \right) - M\ (1-\lambda h)\, L_u\, L_g\ \rho - \eta(h) \geq$$

$$\geq u^h_{\rho,T}(0,x) - M \sum_{i=0}^{\mu-1} \Big((1-\lambda h)^i\, \psi(T-(i+1)h)\, h^2 \Big) - M\ (1-\lambda h)\, L_u\, L_g\, \rho - \eta(h)$$

Then, taking limit when $\rho \to 0$ the last inequality becomes:

- Case $\lambda < 2\, L_g$

$$\int_0^T f(y(s),\, u(s))\, e^{-\lambda s}\, ds \geq u^h_T(0,x) - M\, e^{(2L_g - \lambda)T}\, h \tag{91}$$

- Case $\lambda > 2\, L_g$

$$\int_0^T f(y(s),\, u(s))\, e^{-\lambda s}\, ds \geq u^h_T(0,x) - M\, h \tag{92}$$

• Case $\lambda = 2\,L_g$

$$\int_0^T f(y(s),\, u(s))\, e^{-\lambda s}\, ds \geq u_T^h(0,x) - M\,T\,h \tag{93}$$

By definition (13), from (91)−(93) it follows the thesis (88).

□

The effect of the regularity property (76) on the rate of convergence is established by the following:

Theorem 5.5: *Under assumptions* (10), (11) *and* H_1 *we have*

$$\left|u_T(0,x) - u_T^h(0,x)\right| \leq \begin{cases} M\,h & \text{if } 2\,L_g < \lambda \\ M\,T\,h & \text{if } 2\,L_g = \lambda \\ M\,e^{(2L_g - \lambda)T}\,h & \text{if } 2\,L_g > \lambda \end{cases}$$

The proof is obvious because by (49) and (51) we have:

$$u_T(0,x) = \inf_{\alpha \in \mathcal{A}} J_T(0,x,\alpha(\cdot)) \leq \min_{\alpha \in \mathcal{A}^h} J_T(0,x,\alpha(\cdot)) = u_T^e(0,x) \leq u_T^h(0,x) + C\,\phi(T)\,h$$

where $\phi(T)$ is given by (48).

As the inverse inequality is given by theorem 5.4, we obtain the desired estimate.

□

Taking in mind that

$$\left|\, u(x) - u^h(x) \,\right| \leq M\,e^{-\lambda T} + \left|\, u_T(0,x) - u_T^h(0,x) \,\right|$$

we obtain the following

Theorem 5.6: *Under assumptions* H_1 *we have*

$$|u(x)-u^h(x)| \le \begin{cases} Ch & \textit{if } 2L_g < \lambda \\ Ch^{\gamma} \quad \gamma\in(0,1) & \textit{if } 2L_g = \lambda \\ Ch^{\gamma/2} & \textit{if } 2L_g > \lambda \end{cases} \tag{93}$$

The proof is obtained using the same procedure employed in theorem 5.2.

Remark 5.1: Estimate (93) improves (62) (estimate obtained in the general case without semiconcavity assumptions) in the set $\gamma > 1$, as it is clearly shown in Figure 1.

Remark 5.2: In [2], Capuzzo Dolcetta-Ishii have obtained under semiconcavity hypotheses the estimate:

$$|u^h(0,x) - u(0,x)| \le \begin{cases} Ch & \text{if } 2L_g < \lambda \\ Ch^{(\gamma-1)} & \text{if } L_g < \lambda < 2L_g \end{cases} \tag{94}$$

In consequence, this estimate improves (62) only in the set $3/2 < \gamma < 2$. This behavior is clearly seen in Figure 2. In Figure 3 it is also possible to see that the estimate (93) is better than (94) in the set $1 < \gamma < 2$.

CONCLUSIONS

1) In [2] Capuzzo Dolcetta-Ishii have obtained estimates of type $h^{\gamma/2}$ for the error of approximation due to time discretization of Hamilton-Jacobi-Bellman equation. These estimates, corresponding to the general case, (without semiconcavity assumptions) have been proved using classical arguments of the viscosity solutions field. In this paper we prove the same estimates using convex analysis techniques.

2) Under semiconcavity assumptions it is possible to prove sharper results. In fact, in [2] improvements of the rate $h^{\gamma/2}$ has been presented, again obtained by arguments related to viscosity solution methods. Here we have proved others improvements using techniques of regularization, convex analysis and the introduction of a suitable family of finite horizon problems. They are sharper than those obtained in [2] in the set $1<\gamma<2$.

3) When one want to obtain numerical results it is necessary to discretize equation (1) also with respect to space variables; in particular, using finite element methods. In that way one can compute a fully discrete solution u_k^h. A general procedure with this aim has been established in [5]. Estimates of the approximation error $\| u(x) - u_k^h(x) \|$ can be obtained essentially from estimates (6) and (93). In general the better error it can be expected is of order $(\sqrt{h} + \frac{k}{\sqrt{h}})^{\gamma}$; under semiconcavity assumptions sharper rates of convergence can be obtained (see [6]).

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FIGURE 1

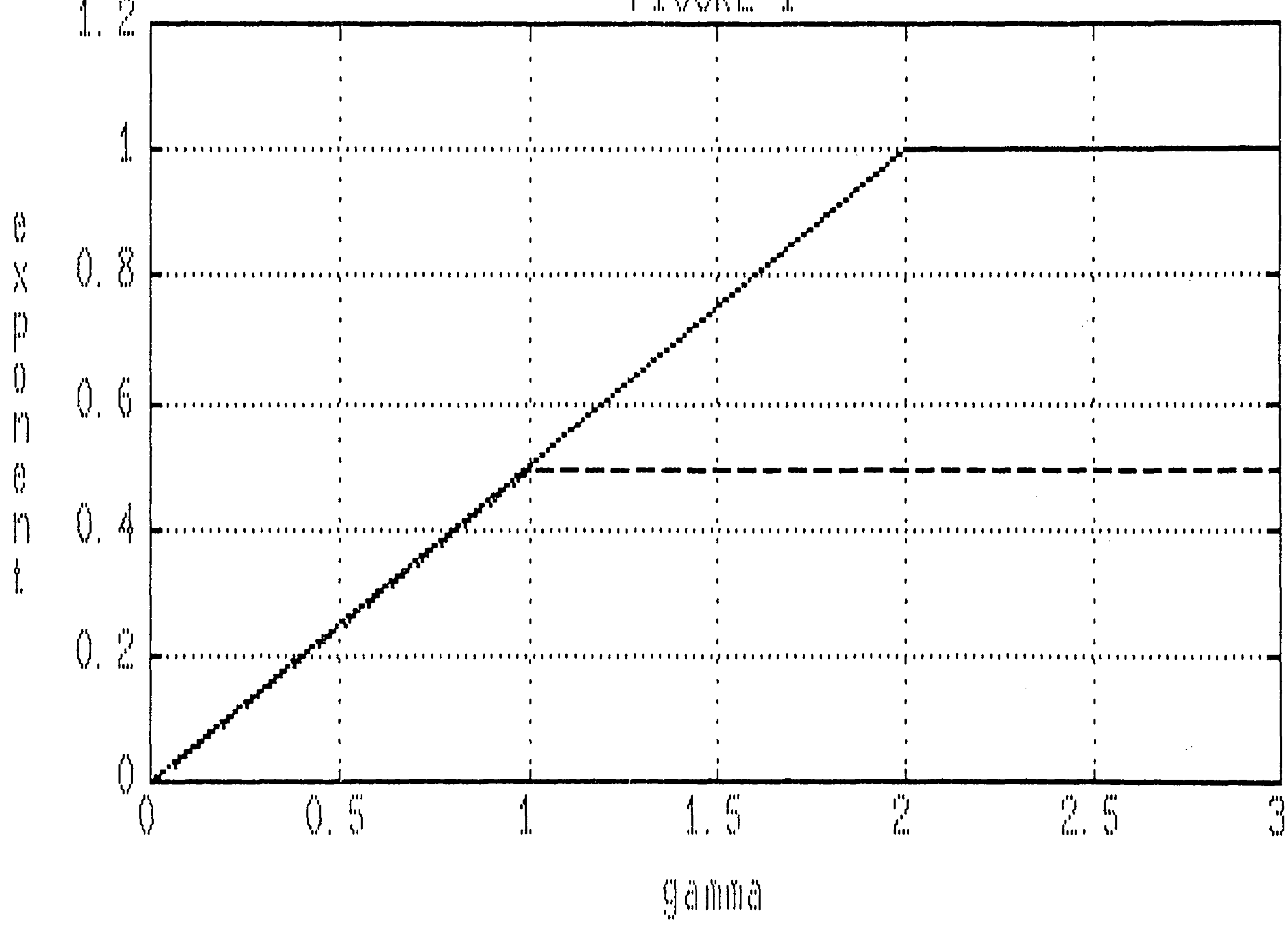


FIGURE 2

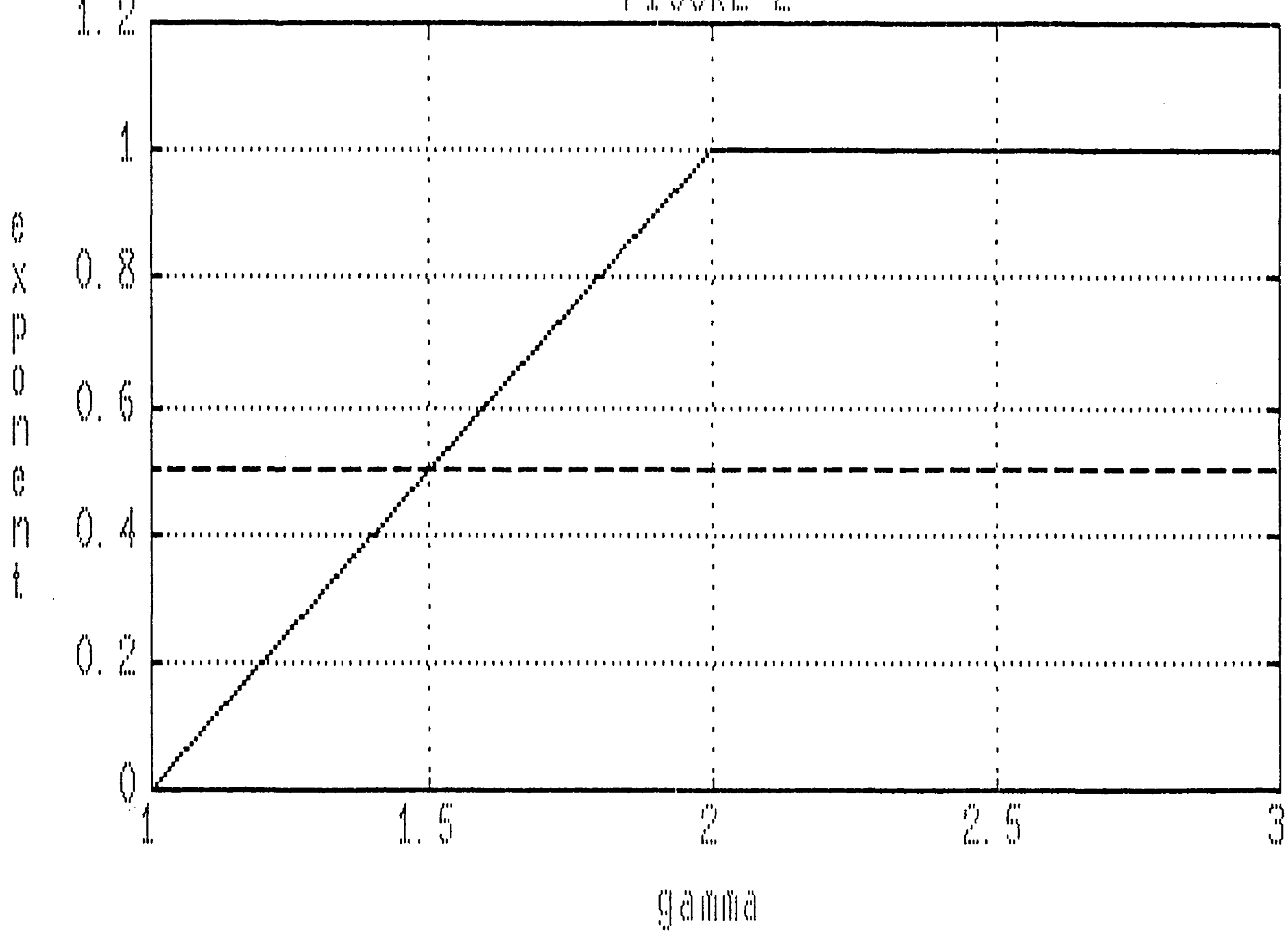
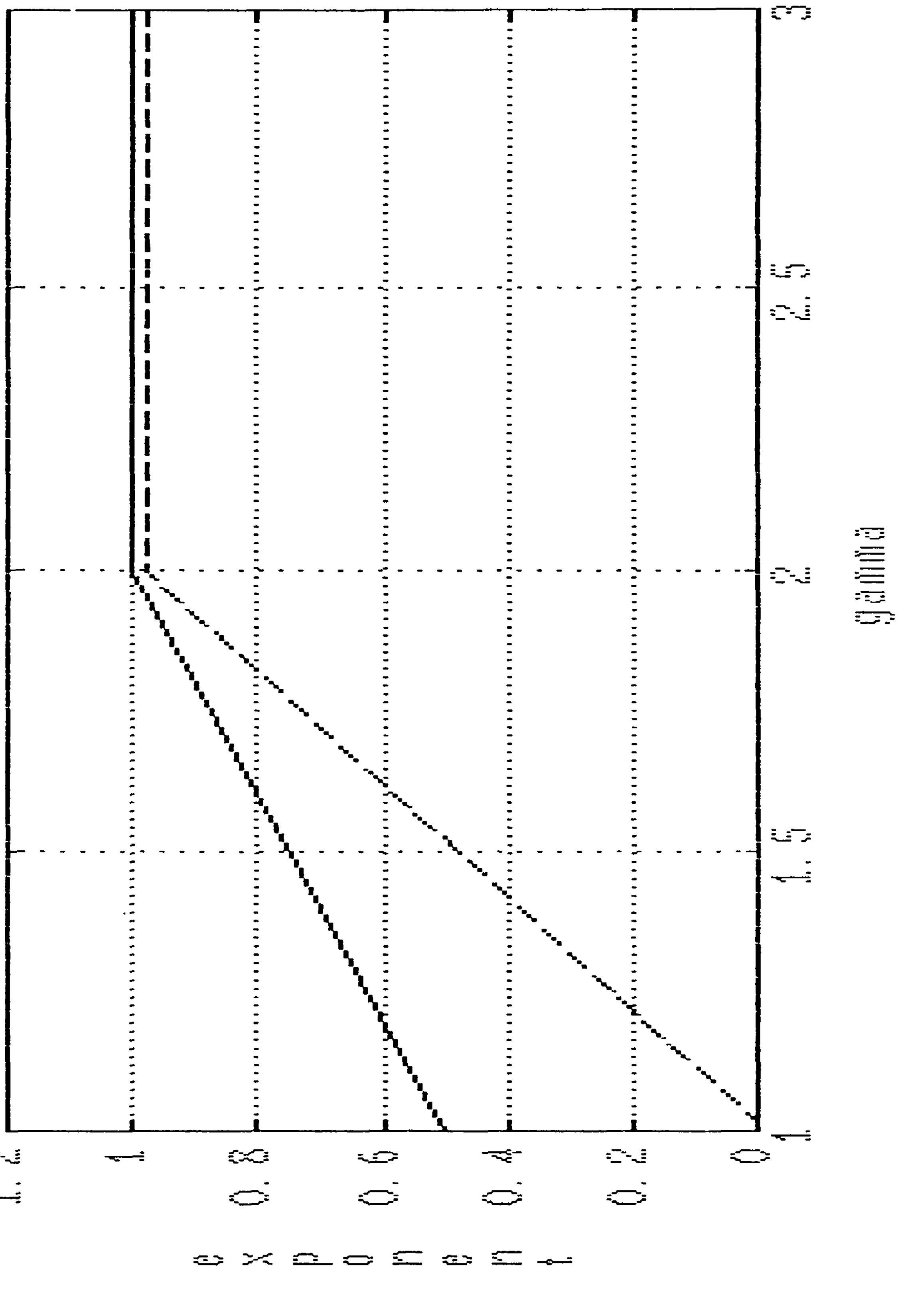


FIGURE 3



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